Using Fibonacci Numbers as an Introduction to Proof

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Introduction

The class began innocently enough with the first ten terms of the Fibonacci sequence written on the board:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55.

This sequence, named after a twelfth-century Italian mathematician, has first and second terms of 1 and each successive term is the sum of the previous two. I casually mentioned to my class that I could instantly find the sum of these terms, 143. My students weren't impressed. I asked them, "What if I let you choose the values for the first two terms and generate the rest using the same method?" With their interest piqued, they chose 2 and 5, generating the sequence:

2, 5, 7, 12, 19, 31, 50, 81, 131, 212

I recorded these terms on the board, as dictated by my students, and quickly announced the sum was 550. A few eyebrows were raised, but most were skeptical. "You were adding them up as you wrote them on the board," accused one student.

Several sequences (and correct sums) later, my students banished me to the hallway, so that I was not allowed to watch them generate their new sequence and record it on the board. As I left the room, they promised the upcoming sequence would be the most difficult yet. Finally, the door opened, I stepped into the room, and was greeted by:

-4, 13, 9, 22, 31, 54, 85, 139, 224, 363.

Without hesitation, I calmly replied, "935" and an outburst of confusion, delight, and consternation ensued. Several students marveled at my supreme computational powers. Others tried to come up with more difficult sequences in an attempt to foil me. A few sat quietly, studying the board, looking for the trick they knew I had to be using. They were hooked. I now had a captive audience of high school freshmen and was ready to embark upon an excursion into patterns, mental computation, and the meaning of proof.

Students and Proof

The classroom discussion described above is the beginning of a lesson designed to introduce students to mathematical proof, an important concept in school mathematics. While many curricula regulate it to geometry classrooms, the National Council of Teachers of Mathematics (2000) believes that proof "should be a consistent part of students' mathematical experience in pre-kindergarten through grade 12" (p. 56). In order to make this vision a reality, teachers must find opportunities for their students to reason and use proof in mathematical contexts other than geometry.

Too often, I have found that high school students write proofs, whether in geometry or precalculus, and do not have the conceptual foundation to justify their actions. Summaries of research (McCrone & Martin, 2004; O'Daffer & Thornquist, 1993), support this belief, suggesting that students have difficulty discerning between examples and proof, determining when proof is necessary, and believing that deductive proofs are more than merely partial evidence that a conjecture is true. The purpose of the lesson described here is to introduce and address these difficulties.

Finding the Pattern

As any reputable magician will not reveal the secrets behind a trick or illusion, I did not reveal the key to my seemingly astounding computational powers. Instead, I gave them time to examine the data (see Table 1). Working alone or in small groups, they produced several conjectures regarding the terms of the sequence, methods for adding them quickly, and their relationship to the sum. In their discussion, a conjecture was abandoned if it didn't work (usually when a counterexample was discovered) or if the method used became too complicated or time-consuming.

Table 1

Examples of Ten-Term Fibonacci Sequences Used in Class

Ten Terms of a Fibonacci-Generated Sequence										Sum
1	1	2	3	5	8	13	21	34	55	143
2	5	7	12	19	31	50	81	131	212	550
4	9	13	22	35	57	92	149	241	390	1012
-5	-3	-8	-11	-19	-30	-49	-79	-128	-207	-539
-8	4	-4	0	-4	-4	-8	-12	-20	-32	-88
-4	13	9	22	31	53	84	137	221	358	924

One student, who was looking at relationships between individual terms and the sum, directed the class to 50, the seventh term of one of the sequences and the sum of that sequence, 550. From these numbers, the class produced the following conjecture: The sum of the first ten terms of any generalized Fibonacci sequence equals its seventh term multiplied by 11. After testing each of the sequences on the board (and a few of their own), my students are convinced that this relationship must be true.

Discussion

Before contemplating the validity of their conjecture, I feel it is important to ask my students how I could multiply larger numbers by eleven quickly and accurately. This question prompts a brief review of the multiplication algorithm and place value, as well as a discussion of the power of some mental computation strategies. After several examples that afford them the chance to practice multiplying by eleven mentally, the discussion moves towards proof.

To initiate the discussion of proof, I ask, "Is this relationship always true?" Next, and more importantly, I ask every mathematics teacher's favorite question, "Why?" While the answer to the first of these seemingly simple questions is invariably agreed upon as "yes," the second invariably sparks some debate.

More likely than not, at least one student will offer the justification, "It works for every example that we've tried." I reply to responses of this type by asking questions such as, "Can you be sure the relationship is true for every possibility?" and, if so, "How many different examples must be verified before you can say, without any doubt, that the relationship is always true? (Is there an official amount?)" In trying to formulate a reply to these questions, my students often become frustrated, as they have become cognizant of the difference between saying that something is always true and being able to show why. This gap between thought and articulation addresses another of my important questions: "Why do we need to prove that the conjecture is true?"

Though students struggle in coming to terms with questions like these, it is in that struggle that they begin to understand why proof is important in mathematics. As teachers, by facilitating these discussions, we are performing an invaluable service. We are giving our students a foundation for systematic reasoning, "a defining feature of mathematics" (NCTM, 2000, p. 57).

A Proof

To formulate a proof for the class' conjecture, I suggested that they might try an algebraic argument. Since using specific values will not justify all cases of this relationship, I asked if there is a way to express "any number you choose," prompting students to consider variables. As a class, we chose a and b to represent the first two terms of the sequence. My students then expressed the rest of the terms of the sequence in terms of a and b (see Figure 1), formulated a value for the sum of the ten terms, formulated a value for eleven times the seventh term, and showed these values were equivalent (see Figure 2).

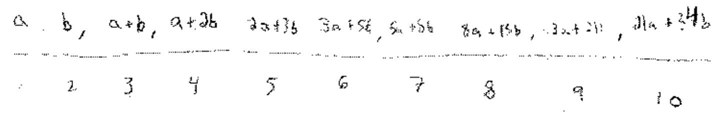

Figure 1. A student's algebraic representation of the first ten terms.

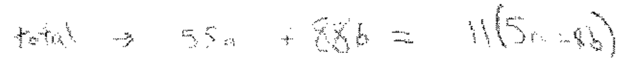

Figure 2: A student's conclusion of the argument.

Once my students completed their work, I asked, "Does this argument prove that this relationship will always be true?" and, most importantly, "Why?" This line of questioning can help in assessing whether or not students believe that a proof is sufficient as a mathematical argument, rather than just more evidence to add support a conjecture. With the conclusion of this discussion, the lesson ended.

Conclusion

Activities of this nature can be structured around mathematical relationships throughout the high school curriculum, not just geometry. Articles by Eric Knuth (2002), Elizabeth Bremigan (2004) and Robert Stanton (2005) in *Mathematics Teacher* provide additional rationale and examples for non-geometric use of proof. By making proof more pervasive in the high school curriculum, we give our students a better opportunity to understand both the content and discipline of mathematics.

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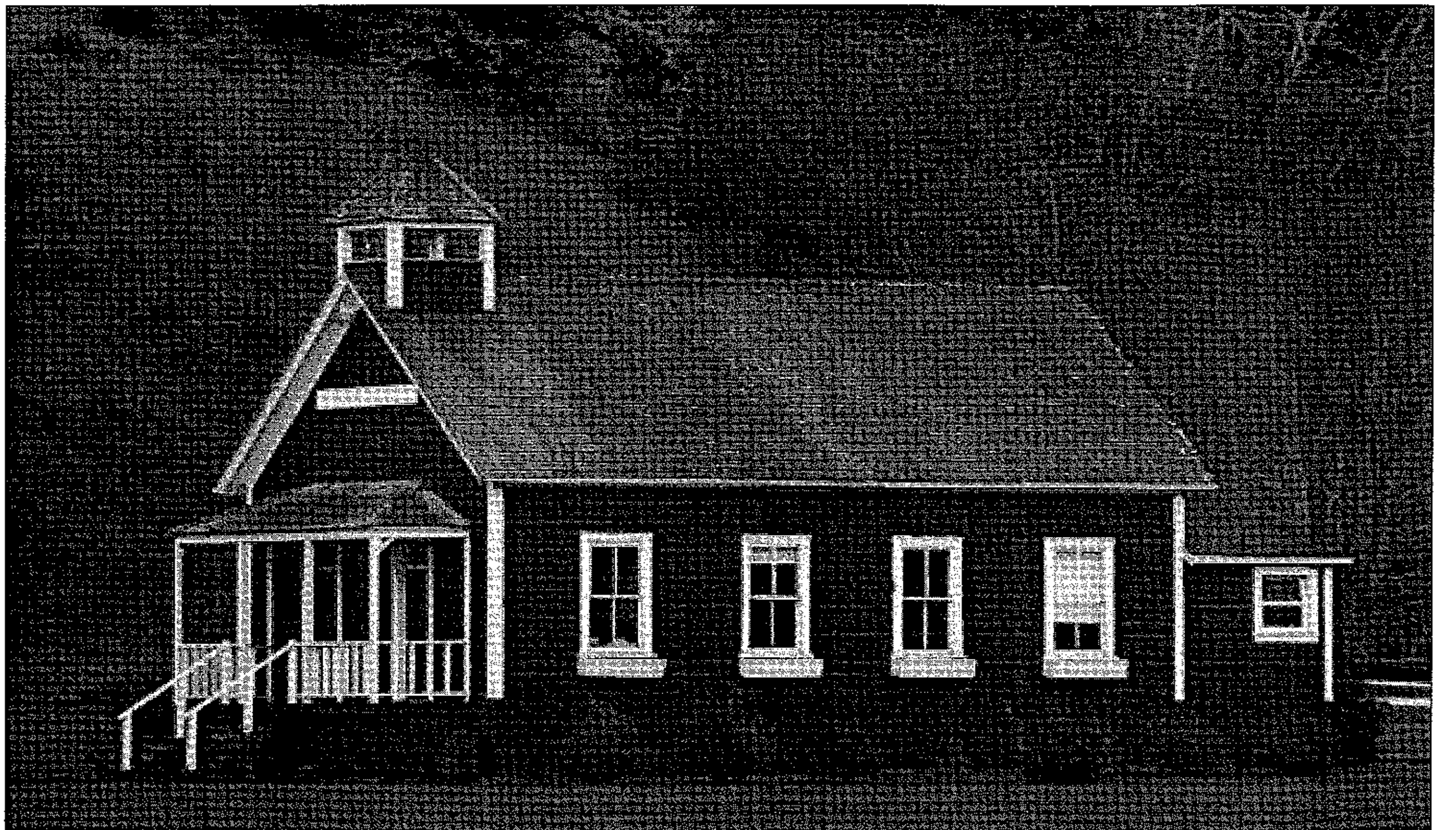
Kyle Schultz is a doctoral student in mathematics education at the University of Georgia. Before returning to his studies, he taught high school mathematics for eight years.



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